INTRODUCTION AND TERMINOLOGY CONCEPT DIFFERENTIAL EQUATION

Differential Equations

An ordinary differential equation is an equation involving an un-know function of one independent variable and one or more of its derivatives.

In most general terms :

$$F(x, y, y', \ldots, y^{(n)}) = 0,$$

Where y is a un-known function of the independent variable x.

Specific example include

Second order

$y'' = y$

Third order

$y''' - yy' = \cos t$

$du/dt + t.u = -u^3$

The order of the differential eq. is the order of the highest order derivative in the equation.

Differential Equations

An ordinary differential equation is an equation involving an un-know function of one independent variable and one or more of its derivatives.

In most general terms :

$$F(x, y, y', \ldots, y^{(n)}) = 0,$$

Where y is a un-known function of the independent variable x.

BY : JHONNY

A solution of such an equation is a function that satisfies the equation for all values of the independent variable in some interval.

- One solution of $y' + y = 0$ is $y=e^{-t}$ for all t in $(-\infty, \infty)$

  $y= e^{-t} \Rightarrow y' + y = -e^{-t} + e^{-t} = 0$

- one solution of $y'' + y = 0$ is $y = \sin x$ for all x in $(-\infty, \infty)$

  $y = \sin x \Rightarrow y'' + y = -\sin x + \sin x = 0$

BY : JHONNY

- The function $y = (1-x)^{-1}$ satisfies the equation

  $y' = y^2$ every where except at $x = 1$.

  $y = (1-x)^{-1} \Rightarrow y' = y^2 = -(1-x)^{-2} (-1) - (1-x)^{-2}$

  $= 0$

  so we have two distinct solutions from the same formula:

  $y = (1/1-x)$ on $(-\infty,1)$ and $y = (1/1-x)$ on $(1, \infty)$

BY : JHONNY

General solution

A differential equation typically has infinitely many solutions that can be described using n parameters, where n is the order of the equation, ex; as an n parameter family.

We call this a general solution of the differential equation.

Equation	General solution
$y'+ y = 0$	$y = Ce^{-t}$
$y'' + y = 0$	$y = A \cos x + B \sin x$
$y' = y^2$	$y = (1/c-t)$

Example:

$y' = e^{-t}$ $\quad$ $y = -e^{-t} + c$

$y''= 0$ $\quad$ $y = c_1 x + c_2$

In principle, it is possible to work “back wards” from a given n parameter general solution to a corresponding n^th^ order DE.

Example: consider the 1- parameter family

$y = \tan^{-1} (x-c)$

$y' = 1/[1+(x-c)^2]$ and $x-c = \tan y$

$y' = 1/[1+\tan^2 y] = 1/\sec^2 y$

$y' = \cos^2 y$

Example: consider the 2-parameter family

$y = t(c_1+c_2 \ln t)$

$y' = c_1+c_2 \ln t + c_2$ and

$y'' = c_2t^{-1}$

$y-ty' = -c_2t \rightarrow c_2 = (y-ty')/t$

$y''=-(y-ty')/t^2$

$t^2y'' - ty' + y = 0$

Initial value problems

A first order initial value problem consist of a first order differential equation along with a given desired value of the solution at some point in its domain.

$$y' = f(x,y); \quad y(x) = y_o$$

Example: given that a general solution of $y'= y^2$ is $y = 1/(c-t')$, where; $y' = y^2$, $y(1) = 2$ and state the interval on which the solution is defined.

$y(1) = 2 \rightarrow 1/c-1 = 2 \rightarrow c = 3/2$

$y = 1/(3/2-t) = 2/(3-2t)$ for $-\infty < t < 3/2$

A second order initial value problem consist of a second order differential equation along with a given desired value of the solution and its first derivative at some point in its domain.

$y'' = f(x, y, y'),\ \ y(x_0) = y_0$

$y'(x_0) = m_0$

Example: given that a general solution of

$y'' + 4y = 0$ is $y = c_1\cos 2t + c_2\sin 2t$,

solve the initial value problem

$y'' + 4y = 0,\ y(0) = 2$

$y'(0) = -1$

$y = c_1\cos 2t + c_2\sin 2t$

$y' = -2c_1\sin 2t + 2c_2\cos 2t$

$y(0) = 2 \rightarrow c_1\,(1) + c_2\,(0) = 2 \rightarrow c_1 = 2$

$y'(0) = -1 \rightarrow -2c_1(0) + 2c_2(1) = -1 \rightarrow c_2 = -1/2$

$y = 2\cos 2t - \frac{1}{2}\sin 2x$

Linear differential equations

A differential equation is linear if it is linear in the dependent variable and it derivatives.

Linear 1st order : $p_1y' + p_0y = f\ (p_1 \neq 0)$
Linear 2nd order : $p_2y'' + p_1y' + p_0y = f\ (p_2 \neq 0)$
Linear nth order : $p_ny^{(n)} + \ldots + p_1y' + p_0y = f\ (p_n \neq 0)$

When f = 0, such an equation is said to be homogeneous.
Usually we write linear equations in a form in which the leading coefficient is 1:

$y' + py = f, \qquad y'' + py' + qy = f, \ldots$

Equation	Description
$y' + e^{-x} y = \sin x$	1st order, linear , non homogeneous
$ty'' + y' + t^2 y = 0$	2nd order, linear, homogeneous
$ty'' + y' + y^2 = 0$	2nd order, non linear
$x'' = x x'$	2nd order, non linear
$y' y = \sin x$	1st order, non linear
$X'' = 3x - e^{-t} x'$ $x'' + e^{-t} x' - 3x = 0$	2nd order, linear, homogeneous
$u' = \cos u$	1st order, nonlinear

BY : JHONNY

The importance of being linear

Given one or more solutions of a linear, homogeneous differential equation, any linear combination of them is also a solution.

DE	Solution (s)	More solution(s)
$y''=0$	x, 1	$ax+b$
$y'' + y = 0$	Cos t, sin t	A cos t + B sin t
$y' - y = 0$	e^x	Ce^x
$y'' - y = 0$	e, e^{-t}	$C_1e^t + c_2e^{-t}$

BY : JHONNY

Without either linearity or homogeneity, this doesn't work.

DE	Solution(s)	Non-solution(s)
$y''=2$	x^2	$2x^2$
$y'=y^2$	$(1-t)^{-1}$	$2(1-t)^{-1}$

BY : JHONNY

Given any two solutions of a linear differential equation, their difference is a solution of the corresponding homogeneous reduced equation.

DE	Solution	Difference	Homogeneous DE
$y'=2t$	t^2+2, t^2	2	$y'=0$
$y''=2$	t^2+t, t^2+1	$t-1$	$y''=0$
$y''+y=1$	$\cos x+1$, $\sin x+1$	$\cos x - \sin x$	$y''+y=0$

What's happening in general?

Suppose that u and v are solution of

$y' + py = f$, then

$u' + pu = f$

$v' + pv = f$

➔$(u-v)' + p(u-v) = 0$

➔ $u-v$ satisfies $y' + py = 0$

This property of linear differential equations turns out to be useful in solving them. The method is called superposition.
Example: consider the equation $y'+y=t$
it is not difficult to see that one particular solution is the linear function $z(t) = t-1$

Now, let $y(t)$ be any other solution, then;
$(y-(t-1))'+(y-(t-1)) = t - t = 0$

Since the general solution of $y' + y = 0$ is:
$y = ce^{-t}$

$y(t)-(t-1) = ce^{-t}$ for some constant c, and
therefore the general solution of $y'+y=t$ is $y(t) = t-1 + ce^{-t}$

The bottom line is that the general solution of a non homogeneous linear differential equation always has the following from:

y = (any particular solution) + (general solution of the reduced equation)

y = particular + ‘complimentary”

The simplest kind of illustration is an ant differentiation problem.

$y'' = 6x$

$y' = 3x^2 + c_1$

$y = x^3 + c_1x + c_2$

Particular solution

Complementary solution

THANK YOU
To be continue…